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COMMENT

Comment on 'Equilibrium crystal shape of the Potts model at the first-order transition point'

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Abstract
We comment on the paper by Fujimoto (1997 *J. Phys. A: Math. Gen.* **30** 3779) where the exact equilibrium crystal shape (ECS) in the critical Q-state Potts model on the square lattice was calculated, and its equivalence with ECS in the Ising model was established. We confirm these results, giving their alternative derivation applying the transformation properties of the one-particle dispersion relation in the six-vertex model. It is shown that this dispersion relation is identical to that in the Ising model on the square lattice.

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In paper [1] Fujimoto determined the equilibrium crystal shape (ECS) in the Q-state Potts model on the square lattice at the first-order transition temperature. Fujimoto claimed [1, 2] that ECS is universal for a wide class of models, including the eight-vertex model [2, 3] and the Ising model on the square lattice [4]. The origin of this universality is still not well understood.

The subject of this comment is to show that the one-particle spectrum in the six-vertex model (which is known to be equivalent to the critical Q-state Potts model) and in the Ising model on the square lattice is same. So, one can say that the universality of ECS in different square-lattice models reflects the universality in the one-particle dispersion relation.

Consider the Q-state Potts model ($Q > 4$) on the square lattice, shown in figure 1. The sites of the Potts lattice are depicted by open circles. The model Hamiltonian is given by

$$\beta E = -K_1 \sum_{j,l} \delta(\sigma_{j,l+1} - \sigma_{j,l}) - K_2 \sum_{j,l} \delta(\sigma_{j+1,l} - \sigma_{j,l}). \tag{1}$$

Here we use the same notation as in [1], constants K_1, K_2 obey the critical temperature condition [3]: $(\exp K_1 - 1)(\exp K_2 - 1) = Q$. Let ψ be a vector associated with a horizontal row of the Potts lattice, $\psi = \psi(\ldots, \sigma_1, \sigma_2, \ldots, \sigma_l, \ldots)$. Denote by T_1 and T_2 the two shift operators

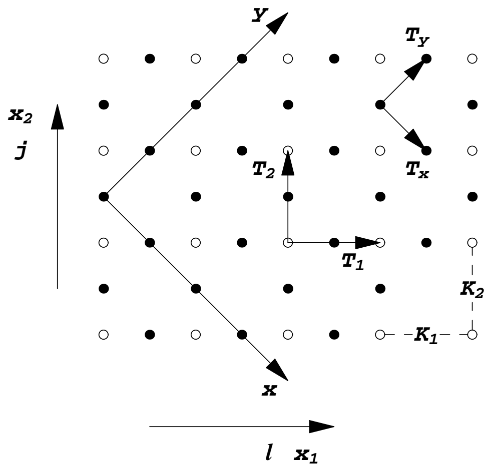

Figure 1. The Potts model and the associated six-vertex model square lattices. The Potts model sites and two-spin interactions are shown by open circles and by broken lines. Indices l and j enumerate the rows and columns of the Potts lattice. The sites of the six-vertex model are shown by full circles.

[1, 2], shown schematically in figure 1. Operator T_2 is the row-to-row transfer matrix of the Potts model, and T_1 acts on the vector ψ as

$$T_1\psi(\ldots,\sigma_1,\sigma_2,\ldots,\sigma_l,\ldots) = \psi(\ldots,\sigma_2,\sigma_3,\ldots,\sigma_{l+1},\ldots). \tag{2}$$

It is well known that the critical Potts model is equivalent to the six-vertex model [3, 5] and to the interaction round a face (IRF) model [1, 6]. Consider for definiteness the associated six-vertex model. Its sites are shown by full circles in figure 1. One can define two shift operators T_x and T_y corresponding to translations along the directions x and y. Note that operator T_y is the row-to-row transfer matrix of the six-vertex model. As is clear from figure 1, these shift operators are related to T_1, T_2 by the equations

$$T_1 \cong T_x T_y \qquad T_2 \cong T_x^{-1} T_y. \tag{3}$$

Consider operator $B(\theta)$, generating a one-particle excitation from the ground state of the six-vertex model. This operator transforms under translations T_x, T_y as

$$T_x B(z) T_x^{-1} = \exp[-\mathrm{i}p_x(z)]B(z) \qquad T_y B(z) T_y^{-1} = \exp[-\mathrm{i}p_y(z)]B(z) \tag{4}$$

where equations

$$\begin{aligned} \exp[-\mathrm{i}p_x(z)] &= \sqrt{k}\,\mathrm{sn}\left(z - \frac{\mathrm{i}K'}{2}\right) \\ \exp[-\mathrm{i}p_y(z)] \equiv \exp[-\omega(z)] &= \sqrt{k}\,\mathrm{sn}\left(z - \frac{2\mathrm{i}Ku}{\pi}\right) \end{aligned} \tag{5}$$

define in the parametric form the dispersion relation $\omega(p_x)$ of the one-particle excitations in the six-vertex model. Here sn is the Jacobian sn function to the modulus k, $0 < k < 1$, the quarter periods K, K' satisfy $K'/K = 2\lambda/\pi$. Parameters λ and u are related to the original parameters of the Potts model by [1]:

$$\sqrt{Q} = 2\cosh\lambda \qquad \frac{\exp(K_1) - 1}{\sqrt{Q}} = \frac{\sinh\left[\frac{1}{2}\lambda - u\right]}{\sinh\left[\frac{1}{2}\lambda + u\right]}. \tag{6}$$

The elliptic modulus k can be explicitly written as

$$k = \left[1 - \prod_{n=1}^{\infty}\left(\frac{1-q^{2n-1}}{1+q^{2n-1}}\right)^8\right]^{1/2} \tag{7}$$

where

$$q = \exp(-2\lambda) = \left(\frac{\sqrt{Q}-\sqrt{Q-4}}{2}\right)^2.$$

The one-particle spectrum (5) can be obtained in the standard manner [7, 8] from the Bethe ansatz solution of the six-vertex model [3]. In the more general case of the eight-vertex model, such calculations have been made by Johnson *et al* [7]. It should be noted that equations (5) also describe the one-particle excitation spectrum in the IRF and the critical Potts model [1, 6]. In the critical Potts model, these particles can be interpreted as the domain walls between the ordered and the disordered phases [9].

Let us transform the dispersion relation (5) as follows. Consider transformation properties of the operator $B(z)$ with respect to the translations along the rows and columns of the original Potts lattice. From (4), (3) one obtains immediately

$$T_1 B(z) T_1^{-1} = \exp[-\mathrm{i}p_1(z)]B(z) \qquad T_2 B(z) T_2^{-1} = \exp[-\mathrm{i}p_2(z)]B(z) \tag{8}$$

where

$$p_1(z) = p_x(z) + p_y(z) \qquad p_2(z) = -p_x(z) + p_y(z). \tag{9}$$

By using (5) and applying the well-known properties of the Jacobian elliptic functions, one can remove parameter z from (9):

$$a_1 \cos p_1 + a_2 \cos p_2 = 1 \tag{10}$$

where

$$a_1 = [(1+\rho^2)(1+k^2\rho^2)]^{-1/2} \tag{11}$$

$$a_2 = k\rho^2[(1+\rho^2)(1+k^2\rho^2)]^{-1/2} \tag{12}$$

$$\rho = -\mathrm{i}\,\mathrm{sn}\left(\frac{2\mathrm{i}uK}{\pi} + \frac{\mathrm{i}K'}{2}\right). \tag{13}$$

Since p_1 and $\epsilon \equiv \mathrm{i}p_2$ define the particle quasimomentum and energy, respectively, equations (10)–(12) give a compact form of the dispersion relation for one-particle excitations in the critical Potts model. The latter is exactly the same as the excitation spectrum in the Ising model on the square lattice first obtained by Onsager [10]. The only difference is in the meaning of parameters k and ρ. In the Potts model, they are given by equations (7) and (13), whereas in the Ising model we have

$$k = [\sinh(2K_1)\sinh(2K_2)]^{-1} \qquad \rho = \sinh(2K_2). \tag{14}$$

Substituting $p_1 \to \mathrm{i}x_2$, $p_2 \to \mathrm{i}x_1$ into the dispersion relation (10) yields [11, 12] the ECS

$$a_1 \cosh x_2 + a_2 \cosh x_1 = 1 \tag{15}$$

where x_1, x_2 denote the Cartesian coordinates in the equilibrium crystal boundary. This shape is the same in the critical Potts model [1] and in the Ising model [4], if the temperature $T < T_c$ and the anisotropy ratio in the Ising model are determined by (14).

It is interesting to note that the dispersion relation (10) is also relevant to the simple Gaussian model on the square lattice, defined by the Hamiltonian

$$\beta E_G = \frac{1}{2}\sum_{j,l}\left\{a\varphi_{j,l}^2 + c_1(\varphi_{j,l+1} - \varphi_{j,l})^2 + c_2(\varphi_{j+1,l} - \varphi_{j,l})^2\right\} \tag{16}$$

where $\varphi_{j,l}$ is a real continuous order parameter. Really, the two-point correlation function in this model can be written as

$$\langle \varphi_{j,l}\varphi_{0,0}\rangle = \frac{1}{a + 2c_1 + 2c_2}\int_{-\pi}^{\pi} \frac{\mathrm{d}p_1\,\mathrm{d}p_2}{(2\pi)^2}\frac{\exp(\mathrm{i}p_1 l + \mathrm{i}p_2 j)}{1 - a_1\cos p_1 - a_2\cos p_2} \tag{17}$$

where

$$a_1 = \frac{2c_1}{a + 2c_1 + 2c_2} \qquad a_2 = \frac{2c_2}{a + 2c_1 + 2c_2}.$$

The dispersion relation is determined by the pole of the integrand in (17), i.e. by equation (10). Since the one-particle dispersion relation determines the angular dependence of the correlation length, the latter is the same over a wide class of square-lattice models including the Ising, six-vertex, critical Potts and the simple Gaussian model (16). It is likely that the origin of this universality lies in the symmetry of the square lattice. If this is the case, one could extrapolate this result to higher dimensions, speculating, for example, that the angular dependence of the correlation length in the three-dimensional Ising model on the cubic lattice could be identical to that in the Gaussian model such as (16) on the same cubic lattice.

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